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Remarks about relations between matter properties and geometrical structures in general relativity.

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work in progress

Geography

Moscow, Saint Petersburg

Bologna, Como, Pisa

Paris, Saclay, Montpellier

Szeged

Cologne

Content

Some other topics

1. Transformation phantom - normal scalar field in some cosmological models
2. Relations between classical and quantum dynamics in models with a soft singularity
3. Quantum tunneling, instantons, birth of the universe and general relativity
4. Conclusions and discussion

Introduction

- The general relativity connects the geometrical properties of the spacetime to its matter content.
  The matter tells to the spacetime how to curve itself, the spacetime geometry tells to the matter how to move.
- The cosmological singularities constitute one of the main problems of modern cosmology.
- The discovery of the cosmic acceleration stimulated the development of "exotic" cosmological models of dark energy; some of these models possess the so called soft or sudden singularities characterized by the finite value of the radius of the universe and its Hubble parameter.

- “Traditional” or “hard” singularities are associated with the zero volume of the universe (or of its scale factor), and with infinite values of the Hubble parameter, of the energy density and of the pressure –Big Bang and Big Crunch
- In some models interplay between the geometry and the matter forces the matter to change some of its basic properties, such as equation of state for fluids and even the form of the Lagrangian.
- Tachyons (Born-Infeld fields) is a natural candidate for a dark energy
- The toy tachyon model, proposed in 2004 has two particular features:
  Tachyon field transforms itself into a pseudo-tachyon field,
  The evolution of the universe can encounter a new type of singularity - the Big Brake singularity.

- The Big Brake singularity is a particular type of the so called "soft" cosmological singularities - the radius of the universe is finite, the velocity of expansion is equal to zero, the deceleration is infinite.
- The predictions of the model do not contradict observational data on supenovae of the type Ia (2009,2010)
- The Big Brake singularity is a particular one - it is possible to cross it (2010)

- Open questions: other soft singularities - is it possible to cross them ?

- What can tell us the Quantum cosmology on the Big Brake singularity and other soft singularities ?

- What is more important: matter or geometry ?

Description of the tachyon model

The flat Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2$$

The tachyon Lagrange density

$$L = -V(T)\sqrt{1 - \dot{T}^2}$$

The energy density

$$\rho = \frac{V(T)}{\sqrt{1 - \dot{T}^2}}$$

The pressure

$$p = -V(T)\sqrt{1 - \dot{T}^2}$$

The Friedmann equation

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \rho$$

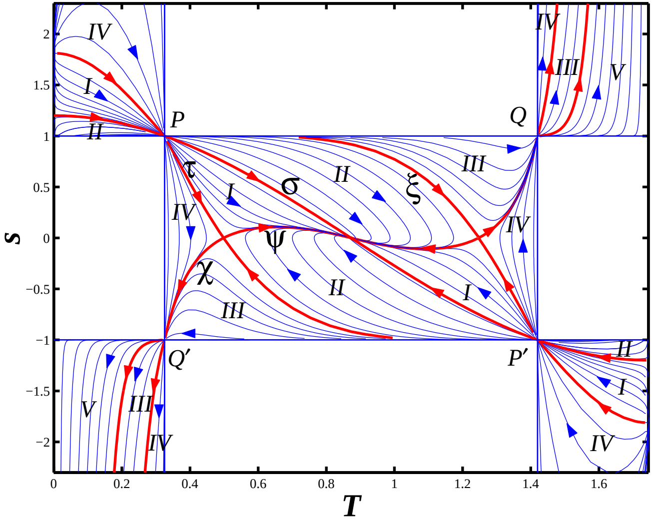
The equation of motion for the tachyon field

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V} = 0$$

In our model

$$V(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]} \times\sqrt{1-(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]}\,,$$

where k and $\Lambda > 0$ are the parameters of the model. The case $k > 0$ is more interesting.



Phase portrait of the model for a positive k.

Some trajectories (cosmological evolutions) finish in the infinite de Sitter expansion. In other trajectories the tachyon field transforms into the pseudotachyon field with the Lagrange density, energy density and positive pressure.

$$L = W(T)\sqrt{\dot{T}^2 - 1},$$

$$\rho = \frac{W(T)}{\sqrt{\dot{T}^2 - 1}},$$

$$p = W(T)\sqrt{\dot{T}^2 - 1},$$

$$W(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\ T\right]}$$

$$\times\sqrt{(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\,T - 1\right]}$$

What happens with the Universe after the transformation of the tachyon into the pseudotachyon ?

It encounters the Big Brake cosmological singularity.

The Big Brake cosmological singularity and other soft singularities

$$t \to t_{BB} < \infty$$
$$a(t \to t_{BB}) \to a_{BB} < \infty$$
$$\dot{a}(t \to t_{BB}) \to 0$$
$$\ddot{a}(t \to t_{BB}) \to -\infty$$
$$R(t \to t_{BB}) \to +\infty$$
$$T(t \to t_{BB}) \to T_{BB}, \ |T_{BB}| < \infty$$
$$|\dot{T}(t \to t_{BB})| \to \infty$$
$$\rho(t \to t_{BB}) \to 0$$
$$p(t \to t_{BB}) \to +\infty$$

If $\dot{a}(t_{BB}) \neq 0$ it is more general soft singularity.

Crossing the Big Brake singularity and the future of the universe

At the Big Brake singularity the equations for geodesics are regular, because the Christoffel symbols are regular (moreover, they are equal to zero).

Is it possible to cross the Big Brake ?

Let us study the regime of approaching the Big Brake.

Analyzing the equations of motion we find that approaching the Big Brake singularity the tachyon field behaves as

$$T = T_{BB} + \left(\frac{4}{3W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{1/3}.$$

Its time derivative $s \equiv \dot{T}$ behaves as

$$s = -\left(\frac{4}{81W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{-2/3},$$

the cosmological radius is

$$a = a_{BB} - \frac{3}{4}a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{4/3},$$

its time derivative is

$$\dot{a} = a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{1/3}$$

and the Hubble variable is

$$H = \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3}(t_{BB} - t)^{1/3}.$$

All these expressions can be continued in the region where $t > t_{BB}$,which amounts to crossing the Big Brake singularity. Only the expression for s is singular at $t = t_{BB}$ but this singularity is integrable and not dangerous.

Once reaching the Big Brake, it is impossible for the system to stay there because of the infinite deceleration, which eventually leads to the decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius and Hubble variable change their signs. The expansion is then followed by a contraction, culminating in the Big Crunch singularity.

The paradox of the soft singularity crossing in the model with the anti-Chaplygin gas and dust

One of the simplest cosmological models revealing the Big Brake singularity is the model based on the anti-Chaplygin gas with an equation of state

$$p = \frac{A}{\rho}, \quad A > 0$$

Such an equation of state arises in the theory of wiggy strings (B. Carter, 1989, A. Vilenkin, 1990).

$$\rho(a) = \sqrt{\frac{B}{a^6} - A}$$

At $a = a_* = \left(\frac{B}{A}\right)^{1/6}$ the universe encounters the Big Brake singularity.

The anti-Chaplygin gas plus dust

The energy density and the pressure are

$$\rho(a) = \sqrt{\frac{B}{a^6} - A} + \frac{M}{a^3}, \quad p(a) = \frac{A}{\sqrt{\frac{B}{a^6} - A}}.$$

Due to the dust component, the Hubble parameter has a non-zero value at the encounter with the singularity, therefore the dust implies further expansion. With continued expansion however, the energy density and the pressure of the anti-Chaplygin gas would become ill-defined.

We solve the paradox by redefining the anti-Chaplygin gas in a distributional sense. Then a contraction could follow the expansion phase at the singularity at the price of a jump in the Hubble parameter. Although such an abrupt change is not common in any cosmological evolution, we explicitly show that the set of Friedmann, Raychaudhuri and continuity equations are all obeyed both at the singularity and in its vicinity.
The jump in the Hubble parameter

$$H \to -H$$

leaves intact the first Friedmann equation $H^2 = \rho$, the continuity equations and the equations of state, however, it breaks the validity of the second Friedmann (Raychaudhuri) equation $\dot{H} = -\frac{3}{2}(\rho + p)$.

$$
\begin{aligned}
H(t) &= H_S sgn(t_S - t) \\
&\quad + \sqrt{\frac{3A}{2H_S a_S^4}} sgn(t_S - t)\sqrt{|t_S - t|}\ ,
\end{aligned}
$$

$$
\dot{H} = -2H_S\delta(t_S - t) - \sqrt{\frac{3A}{8H_S a_S^4}}\frac{sgn(t_S - t)}{\sqrt{|t_S - t|}}\ .
$$

To restore the validity of the Raychaudhuri equation we add a singular δ -term to the pressure of the anti-Chaplygin gas

$$
p = \sqrt{\frac{A}{6H_S|t_S - t|}} + \frac{4}{3}H_S\delta(t_S - t).
$$

To preserve the equation of state we also modify the expression for its energy density:

$$
\rho = \frac{A}{\sqrt{\frac{A}{6H_S|t_S - t|}} + \frac{4}{3}H_S\delta(t_S - t)}\ .
$$

In order to prove that p and ρ represent a self-consistent solution of the system of cosmological equations, we used the following distributional identities:

$$
\begin{aligned}
\left[sgn\left(\tau\right)g\left(\left|\tau\right|\right)\right]\delta\left(\tau\right) &= 0\ , \\
\left[f\left(\tau\right)+C\delta\left(\tau\right)\right]^{-1} &= f^{-1}\left(\tau\right)\ , \\
\frac{d}{d\tau}\left[f\left(\tau\right)+C\delta\left(\tau\right)\right]^{-1} &= \frac{d}{d\tau}f^{-1}\left(\tau\right)\ .
\end{aligned}
$$

Change of the equation of state at soft singularity crossings

The abrupt transition from the expansion to the contraction of the universe does not look natural. There is an alternative/complementary way of resolving the paradox.

One can tray to change the equation of state of the anti-Chaplygin gas at passing the soft singularity.

There is some analogy between the transition from an expansion to a contraction of a universe and an absolutely elastic bounce of a ball from a wall in classical mechanics. There is also an abrupt change of the direction of the velocity (momentum).

However, we know that really the velocity is changed continuously due to the deformation of the ball and of the wall.

In our model the anti-Chaplygin gas at the moment of encountering the soft singularity when

$$\sqrt{\frac{B}{a^6} - A} = 0$$

transforms itself into the Chaplygin gas with negative energy density:

$$\sqrt{\frac{B}{a^6} - A} \to -\sqrt{A - \frac{B}{a^6}}.$$

The pressure remains positive, expansion continues. The spacetime geometry remains continuous.
The expansion stops at $a = a_0$, where

$$\frac{M}{a_0^3} - \sqrt{A - \frac{B}{a_0^6}} = 0.$$

Then the contraction of the universe begins.

At the moment when the energy density of the Chaplygin gas becomes equal to zero

(again a soft singularity), the Chaplygin gas transforms itself into the anti-Chaplygin gas

and the contraction continues to culminate in the encounter with the Big Crunch singularity $a = 0$.

Transformation phantom - normal scalar field in some cosmological models

Some cosmological observations point out the the present cosmic acceleration is such that

$$w = \frac{p}{\rho} < -1.$$

Phantom matter.
Phantom scalar field :

$$L = -\frac{\dot{\phi}^2}{2} - V(\phi).$$

Standard scalar field:

$$L = \frac{\dot{\phi}^2}{2} - V(\phi).$$

Some observations tell that it was a moment when $w+1$ has changed the sign.
Phantom divide line crossing

Is it possible to have this phenomenon in the model with one scalar field - the transformations between phantom scalar field and normal scalar field ?

Yes ! If two conditions are satisfied:

The potential $V(\phi)$ has a cusp.
The initial conditions are fixed in such a way that the scalar (or phantom scalar) field arrives at the cusp with the vanishing velocity $\dot{\phi}$.

Relations between classical and quantum dynamics in models with a soft singularity

There is an old hypothesis that the classical cosmological singularities disappear in the quantum theory.

That means that introducing a quantum state (wave function) of the universe one can calculate quantum probabilities of realization of different classical configurations and to see that these probabilities disappear for those configurations of parameters, which correspond to classical singularities.

We have studied three cosmological models with soft singularities: the tachyon model with trigonometrical potential, the tachyon model with constant potential and minimally coupled scalar field model with the Lagrangian

$$L = \frac{\dot{\phi}^2}{2} - \frac{V_0}{\phi}, \;\; V_0 > 0.$$

In all three cases the effect of quantum avoidance of singularities is absent for the classically traversable soft singularities and is present for "hard" Big Bang and Big Crunch singularities.

Quantum tunneling, instantons, birth of the universe and general relativity

In the modern cosmology the notions of the wave function of the universe of the quantum birth of the universe and of the quantum tunneling are connected.

The link between them is constituted by the instantons - the solutions of Euclidean Einstein equations.

Then one should carry out some kind of analytical continuation from the instantons to the spacetimes with the Lorentzian signature - "birth of the universe".

Usually, the matter presented in these instantons behaves approximately like a cosmological constant.

If we consider the matter consisting of two components - (quasi)-cosmological constant and the radiation, which can be represented by some set of conformal fields, then:

1. The quantum state of the universe is not a pure quantum state, described by the wave function of the universe but a mixed quantum state, described by the cosmological density matrix.
2. One obtains a system of two coupled equations, whose solution gives some restrictions on the matter content of the universe.

The modified Friedmann equation

$$\frac{\dot{a}^2}{a^2} + B\left(\frac{1}{2}\frac{\dot{a}^4}{a^4} - \frac{\dot{a}^2}{a^4}\right) = \frac{1}{a^2} - H^2 - \frac{C}{a^4},$$

The amount of radiation constant C is given by the bootstrap equation

$$m_P^2 C = m_P^2 \frac{B}{2} + \frac{dF(\eta_0)}{d\eta_0} \equiv m_P^2 \frac{B}{2} + \sum_{n=1}^{\infty} \frac{n^3}{e^{n\eta_0} - 1}.$$

Full conformal time

$$\eta_0 = 2\int_{\tau_-}^{\tau_+} \frac{d\tau}{a(\tau)}.$$

Conclusions and discussion

- The general relativity contains a lot of surprises concerning relations between the matter and geometry. It is enough to take it seriously.
- The things become even more surprising when we combine the general relativity with quantum theory.